MATH 245 S25, Exam 3 Solutions

1. Carefully define the following terms: intersection, subset.

   Let A, B be arbitrary sets. Their intersection is the set given by $\{x : x \in A \wedge x \in B\}$. Again let A, B be sets. The proposition (or statement) "A is a subset of B" means that every element of A is an element of B.

2. Carefully define the following terms: disjoint, reflexive.

   The pair of sets A, B are disjoint if $A \cap B = \emptyset$. Let R be a relation on some set S. We say that R is reflexive if $\forall x \in S, (x, x) \in R$.

3. Let $R = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z},\ x = 15y\}$, $S = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z},\ x = 30y\}$, $T = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z},\ x = 5y\}$. Prove or disprove that $R \cup S \subseteq T$.

   The statement is true. Let $x \in R \cup S$. Then $x \in R \vee x \in S$. We have two cases.
   $x \in R$: Then $\exists y \in \mathbb{Z},\ x = 15y$. Now $x = 5(3y)$, and $3y \in \mathbb{Z}$, so $x \in T$.
   $x \in S$: Then $\exists y \in \mathbb{Z},\ x = 30y$. Now $x = 5(6y)$, and $6y \in \mathbb{Z}$, so $x \in T$.
   In both cases, $x \in T$.

   A common student error was quantifying y incorrectly, or not at all. Note that there are two different y's: one that lives entirely inside the $x \in R$ case, and one that lives entirely inside the $x \in S$ case. Each needs its own quantifier.

4. Prove or disprove: For any sets R, S, we have $S \subseteq (R\Delta S)\Delta R$.

   The statement is true. Let R, S be arbitrary sets, and $x \in S$. We now have two cases.
   $x \in R$: Since $x \in R$ and $x \in S$, $x \notin R\Delta S$. But $x \in R$, so $x \in (R\Delta S)\Delta R$.
   $x \notin R$: Since $x \notin R$ and $x \in S$, $x \in R\Delta S$. But $x \notin R$, so $x \in (R\Delta S)\Delta R$.
   In both cases, $x \in (R\Delta S)\Delta R$.

   NOTE 1: My proof omits some details of each case, due to space limitations. If you wish (and some of you did), you can leave those details in. For example, in the case $x \notin R$: Since $x \notin R$ and $x \in S$, by conjunction $x \notin R \wedge x \in S$, and by addition $(x \in R \wedge x \notin S) \vee (x \notin R \wedge x \in S)$, so $x \in R\Delta S$. By conjunction, $x \in R\Delta S \wedge x \notin R$ and by addition $(x \in R\Delta S \wedge x \notin R) \vee (x \notin R\Delta S \wedge x \in R)$ so $x \in (R\Delta S)\Delta R$.

   NOTE 2: In grading this, I gave three easy opportunities for partial credit: quantifying R/S, proof structure of $\subseteq$, definition of Δ. You could get 8/10 from this alone.

   NOTE 3: We can use commutativity to get $(R\Delta S)\Delta R = (S\Delta R)\Delta R$, and associativity to get $(S\Delta R)\Delta R = S\Delta(R\Delta R)$. If we had a theorem that said $R\Delta R = \emptyset$ and another one that said $S\Delta\emptyset = S$, we would be more than done. Alas, we don't have these theorems, they are from the homework.

5. Let S, U be sets with $S \subseteq U$. Prove that $S \cap S^c = \emptyset$.
   Note: This is part of Theorem 9.2. Do not use the theorem to prove itself!

   We first suppose there is some $x \in S \cap S^c$. Then $x \in S \wedge x \in S^c$. By simplification twice, $x \in S$ and $x \in S^c$. Hence $x \in U \wedge x \notin S$, and by simplification $x \notin S$. So we have $x \in S$ and $x \notin S$, a contradiction! So $x \in S \cap S^c$ is false, which makes $x \in S \cap S^c \to x \in \emptyset$ true vacuously. This proves $S \cap S^c \subseteq \emptyset$

   For the (optional) direction of $\emptyset \subseteq S \cap S^c$, note that $x \in \emptyset \to x \in S \cap S^c$ is also true vacuously.

   ALTERNATIVE: Suppose that $|S \cap S^c| \geq 1$ (or $S \cap S^c \neq \emptyset$). Then there is some $x \in S \cap S^c$. Proceed as before to get a contradiction, so $|S \cap S^c| = 0$ and hence $S \cap S^c = \emptyset$.

   Many of you managed to get a contradiction, but then struggled to complete the proof. A contradiction is not a get-out-of-jail-free card, proving whatever you want to be true! It proves that the one specific hypothesis you made at the beginning (you did make one, didn't you?) must be false, that is all.

6. Prove or disprove: For all sets R, S, if $R \times S = S \times S$, then $R = S$.
The statement is false, and requires a counterexample. We need to choose $S = \emptyset$, and R to be something else, such as $R = \{4\}$. Now $R \times S = \emptyset = S \times S$. Also, $R \neq S$, since $4 \in R$ and $4 \notin S$.

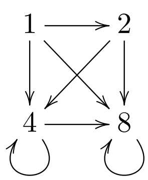
7. Prove or disprove: For all sets S, we must have $|S| \leq |2^S|$.
This was one of your homework problems, but it seems that many of you skipped this one. It requires a pairing between the elements of S and some subset of 2^S. The natural one is: $x \leftrightarrow \{x\}$. Every element of S gets paired with some element of 2^S (i.e. some subset of S, of cardinality 1 in this pairing).
Of course, plenty of elements of 2^S don't get paired, but that's fine, we're only proving $\leq$.

ALTERNATE: If you want to get exotic, this isn't the only pairing that could work. Instead you could have $x \leftrightarrow S \setminus \{x\}$.

8. Prove that relation $R_{diagonal}$, on set $\mathbb{Z}$, is both symmetric and antisymmetric.
Both proofs are very short, so the correct proof structure and every small detail is important.
Symmetric: Let $x, y \in \mathbb{Z}$, and suppose $(x, y) \in R_{diagonal}$. Then $x = y$ (since $R_{diagonal} = \{(a, a) : a \in \mathbb{Z}\}$), so also $(y, x) \in R_{diagonal}$.
Antisymmetric: Let $x, y \in \mathbb{Z}$, and suppose $(x, y) \in R_{diagonal}$ and also $(y, x) \in R_{diagonal}$. Well already from $(x, y) \in R_{diagonal}$ we know that $x = y$.

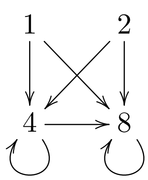
For problems 9-10, we take $S = \{1, 2, 4, 8\}$ and define relation $R = \{(a, b) : 4a | b^2\}$ on S.

9. Draw the digraph for this relation and determine $|R|$.

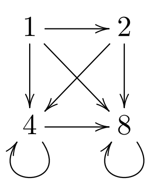


We have $|R| = 8$, since there are eight directed edges.

10. Calculate $R \circ R$ and R^+. Give each in both set and digraph notation.



$R \circ R$ consists of all two-edge paths in R. This turns out to have every edge of R but one, due to all the loops (once you get to 4 or 8, you can stay there). Specifically $R \circ R = \{(1, 4), (1, 8), (2, 4), (2, 8), (4, 4), (4, 8), (8, 8)\}$.



It turns out that $R^+ = R$, by Theorem 10.20 (since R is transitive). So $R^+ = \{(1, 2), (1, 4), (1, 8), (2, 4), (2, 8), (4, 4), (4, 8), (8, 8)\}$.